جامعة بيرزيت
BIRZEIT UNIVERSITY

Key

Name:________________ MATH 332 First Semester 2018/2019

Number:________________ Exam 1

1

(Q#1) (17points) let $f(x) = \pi - |x| \quad , -\pi \le x \le \pi \qquad , f(x+2\pi) = f(x)$

(a) Find Fourier series expansion of $f(x)$, where does it converge?

(c) Can we find the expansion of $F(x) = \int_{-\pi}^{x} f(t)dt$ using the expansion in (1), if so find it.

(d) Can we find f' by using term by term differentiation in (1), if so find it.

1

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{\pi}{2}$$ 2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\Big|_0^{\pi}\right]$$

$\pi - x$ $\cos nx$

-1 $\frac{\sin nx}{n}$

0 $-\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[0 - \frac{(-1)^n}{n^2} - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$ 4

$$= \begin{cases} \frac{4}{\pi n^2} & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases}$$

2

$b_n = 0 \quad , f(x)$ is even

$$\Rightarrow f(x) = \pi - |x| = \frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n^2}\cos nx$$

it converges every where since $f(x)$ is cont

1

(2) since $a_0 \neq 0$, we can find the

expansion of integral of $g(x) = f(x) - a_0$

$$= \pi - |x| - \frac{\pi}{2}$$

$$= \frac{\pi}{2} - |x|$$

2

$$\Rightarrow \int_0^x \left(\frac{\pi}{2} - |s|\right) ds = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n^3} \sin nx$$

2

$c = 0$, since $g(x)$ is even, so $G(x)$ is odd 2

(d) yes, since $f(x)$ is continuous so

$$f'(x) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin nx}{n}$$

2

(Q#2) (15 points) (a) Find the eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0 \quad , \quad 0 < x < \pi$$
$$y(0) = 0 \ , y'(\pi) = 0$$

3

(b) Use the eigenfunctions above to expand $f(x) = x$

$\lambda = 0$

$y'' = 0$

2

$y(x) = ax + b$
$y(0) = 0 = b$
$y(x) = a\cancel{x} \Rightarrow y' = a = 0 \Rightarrow$

} $\lambda = 0$ is not an eigenvalue

$\lambda < 0 \Rightarrow \lambda = -k^2$

$y'' - k^2 y = 0 \Rightarrow r^2 - k^2 = 0 \Rightarrow r = \pm k$

$y(x) = c_1 e^{kx} + c_2 e^{-kx}$

$y(0) = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$

2

$y'(x) = k c_1 e^{kx} - k c_2 e^{-kx}$

$= k c_1 e^{kx} + k c_1 e^{-kx}$

$y'(\pi) = k c_1 (e^{k\pi} + e^{-k\pi})$

$= 2 \underset{\neq 0}{k} c_1 \underbrace{\cosh(k\pi)}_{\neq 0} \Rightarrow c_1 = 0 = c_2$

$\lambda > 0 \Rightarrow \lambda = k^2$

$y'' + k^2 y = 0 \Rightarrow r^2 + k^2 = 0$

5

1 $y(x) = c_1 \cos kx + c_2 \sin kx$

$y(0) = 0 = c_1 \Rightarrow y(x) = c_2 \sin kx$

$y'(x) = k c_2 \cos kx \Rightarrow y'(\pi) = 0$

2 $\Rightarrow \cos k\pi = 0 \Rightarrow k\pi = \frac{(2n-1)\pi}{2}$

1 $k = \frac{2n-1}{2} \Rightarrow$ eigenvalues $\lambda = \left(\frac{2n-1}{2}\right)^2, \ n = 1, 2, \ldots$

1 Eigenfunctions $\phi_n(x) = \sin\left(\frac{2n-1}{2} x\right)$

(1) $$X = \sum_{n=1}^{\infty} \frac{\langle x, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n$$

(4)

$$\langle x, \phi_n \rangle = \int_0^{\pi} x \sin\left(\frac{2n-1}{2} x\right) dx$$

$x \quad \sin\left(\frac{2n-1}{2}\right)x$

$1 \quad -\frac{2}{(2n-1)} \cos\frac{(2n-1)}{2}$

$0 \quad -\frac{4}{(2n-1)} \sin\ldots$

(2)

$$= -\frac{2x}{(2n-1)^2} \cos\left(\frac{2n-1}{2} x\right) - \frac{4}{(2n-1)^2} \sin\left(\frac{2n-1}{2}\right) x \Big|_0^{\pi}$$

$$= -\frac{-4}{(2n-1)^2} (-1)^n = \frac{(-1)^{n+1} 4}{(2n-1)^2}$$

(2) $$\langle \phi_n, \phi_n \rangle = \int_0^{\pi} \sin^2\left(\frac{2n-1}{2} x\right) dx$$

$$\int_0^{\pi} \left(\frac{1 - \cos(2n-1)x}{2}\right) dx$$

$$= \frac{\pi}{2} - \frac{1}{(2n-1)} \sin(2n-1)x \Big|_0^{\pi} = \pi/2$$

$$\Rightarrow (1)\; X = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1)x$$

(3) 5

(a) $H(u-1)e^u = \begin{cases} 0 & u<1 \\ e^u & u\geq 1 \end{cases}$

$$\int_0^x H(u-1)e^u\,du = \begin{cases} 0 & x<1 \\ \int_1^x e^u\,du & x\geq 1 \end{cases} \quad (1)$$

$$= \begin{cases} 0 & x<1 \\ e^x - e & x\geq 1 \end{cases} \quad (1) \quad = (e^x - e)\,H(x-1) \quad (1)$$

(b) Fourier integral of $f(x)$

$$= \int_0^\infty (a(w)\cos wx + b(w)\sin wx)\,dw$$

where

$$a(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx$$

(2)
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}[H(x+1) - H(x-3)]\cos(wx)\,dx$$

$$= \frac{1}{\pi}\int_{-1}^{3}\cos wx\,dx = \frac{1}{\pi}\left.\frac{\sin wx}{w}\right|_{-1}^{3}$$

$$= \frac{1}{\pi w}(\sin 3w + \sin w)$$

$$b(w) = \frac{1}{\pi}\int_{-\infty}^{\infty}(H(x+1) - H(x-3))\sin wx\,dw$$

(2)
$$= \frac{1}{\pi}\int_{-1}^{3}\sin wx\,dw = \frac{-1}{\pi w}\cos wx\Big|_{-1}^{3}$$

$$= \frac{-1}{\pi w}(\cos 3w - \cos w)$$

(2) $$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin 3w + \sin w}{w}\cos(wx) + \frac{\cos w - \cos 3w}{w}\sin wx\right]dw$$

(c) $(g(ax)\cos cx)\hat{}$ (6)

(1) $= \int_{-\infty}^{\infty} g(ax)\cos(cx)\, e^{-i\omega x}\, dx$

(1) $= \int_{-\infty}^{\infty} g(ax)\left(\frac{e^{icx} + e^{-icx}}{2}\right) e^{-i\omega x}\, dx$

(1) $= \frac{1}{2}\int_{-\infty}^{\infty} g(ax)\, e^{-i(w-c)x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} g(ax)\, e^{-i(w+c)x}\, dx$

Let $u = ax$, $du = a\,dx$

i.e. $x = \frac{u}{a}$, $dx = \frac{du}{a}$

(3) $= \frac{1}{2}\int_{-\infty}^{\infty} g(u)\, e^{-i(w-c)\frac{u}{a}}\, \frac{du}{a} + \frac{1}{2}\int_{-\infty}^{\infty} g(u)\, e^{-i(w+c)\frac{u}{a}}\, \frac{du}{a}$

$= \frac{1}{2a}\left[\int_{-\infty}^{\infty} g(u)\, e^{-i\left(\frac{w-c}{a}\right)u}\, du + \int_{-\infty}^{\infty} g(u)\, e^{-i\left(\frac{w+c}{a}\right)u}\, du\right]$

(3) $= \frac{1}{2a}\left[\hat{g}\left(\frac{w-c}{a}\right) + \hat{g}\left(\frac{w+c}{a}\right)\right]$